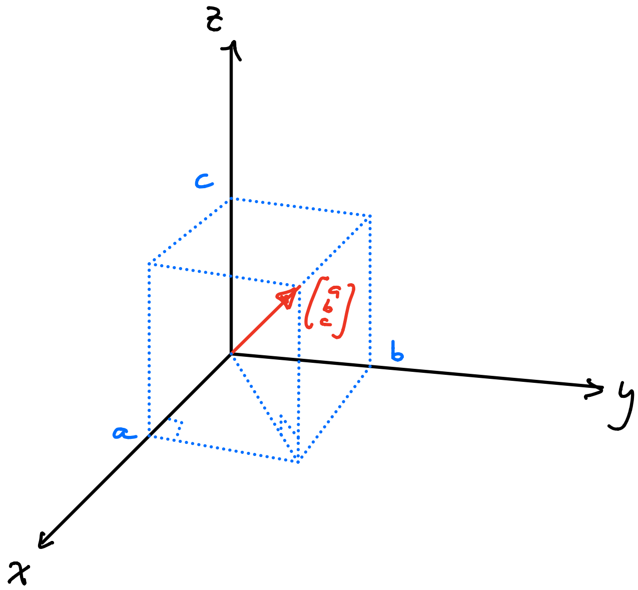
Vectors and lines

Vectors in $\mathbb{R}^3$

Just like in $\mathbb{R}^2$, we can represent a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ geometrically by an arrow from the origin to the point (a, b, c)



The length of a vector $\vec{v}$ is written $\|\vec{v}\|$.

Theorem: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a vector.

1.) $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$

2.) $\vec{v} = \vec{0}$ if and only if $\|\vec{v}\| = \vec{0}$

3.) $\|a\vec{v}\| = |a| \, \|\vec{v}\|$ for scalars a.

4.) $\vec{v} = \vec{w}$ if and only if they have the same direction and length.

Ex: If $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$, then $\|\vec{v}\| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{9} = 3.$

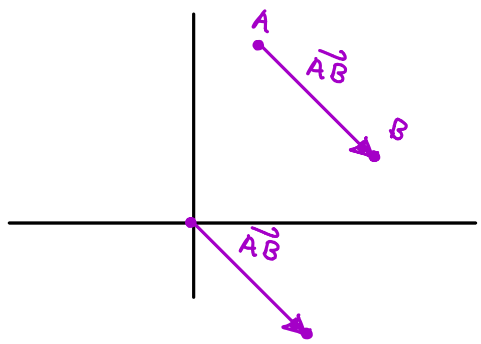
Note: $a\vec{v}$ has the same direction as $\vec{v}$ if $a>0$, and the opposite direction if $a<0$.

So for $\vec{v} \neq \vec{0}$, $\frac{1}{\|\vec{v}\|}\vec{v}$ has length $\frac{\|\vec{v}\|}{\|\vec{v}\|} = 1$, and points in the same direction as $\vec{v}$, called a <u>unit vector</u>.

If A and B are two points in $\mathbb{R}^3$, the vector from A to B is denoted $\overrightarrow{AB}$.

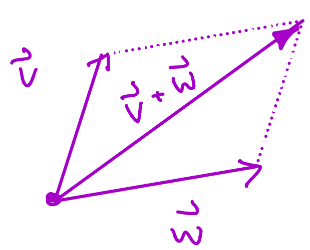
If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, then

$$\overrightarrow{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$$

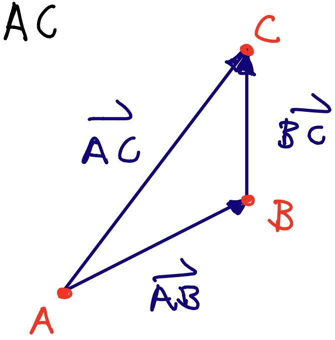


i.e. its coordinates are the ones when it's positioned at the origin.

Parallelogram Law: In the parallelogram determined by $\vec{v}$ and $\vec{w}$, the vector $\vec{v}+\vec{w}$ is the diagonal.



For points A, B, and C, we can write this as
$\vec{AB} + \vec{BC} = \vec{AC}$



So $\vec{AC} - \vec{AB} = \vec{BC}$

i.e. the vector from the tip of $\vec{AB}$ to the tip of $\vec{AC}$.

Ex: If $P_1 = (1,-1,2)$ and $P_2 = (3,0,4)$, then the vector $\vec{P_1P_2}$ is $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, and the distance from P_1 to P_2 is $\sqrt{2^2+1^2+2^2} = 3$.

Thm: Two nonzero vectors are parallel if and only if one is a scalar multiple of the other.

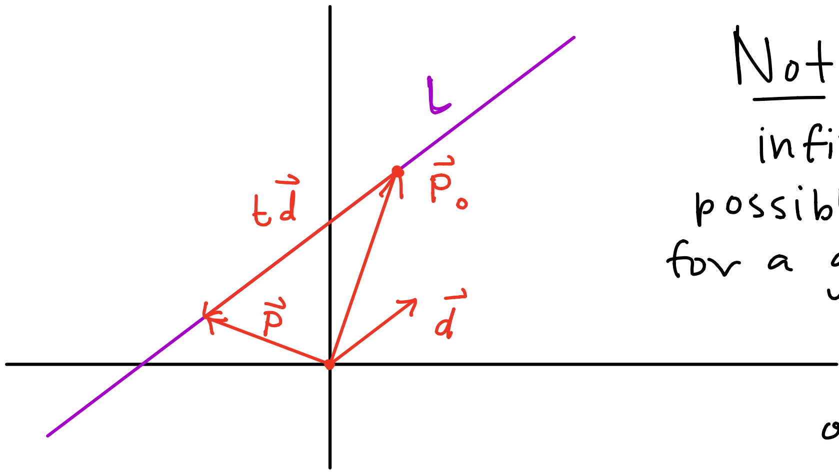
Ex: If $P=(2,-1,4)$, $Q=(3,-1,3)$, $A=(0,2,1)$, $B=(1,3,0)$, are $\vec{PQ}$ and $\vec{AB}$ parallel? The corresponding vectors are

$\begin{bmatrix} 3-2 \\ -1-(-1) \\ 3-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1-0 \\ 3-2 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, respectively, which are not scalar multiples of each other, and thus not parallel.

Lines

Let L be a line in $\mathbb{R}^3$ (or $\mathbb{R}^2$). $\vec{d}$ is a <u>direction vector</u> for L if it is parallel to the line.

e.g. if P, Q are distinct points in L, then $\overrightarrow{PQ}$ is a direction vector for L.



Note: There are infinitely many possible direction vectors for a given line, and they are all scalar multiples of eachother.

If (x_0, y_0, z_0) is a point on L, and $\vec{p}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$,

Then we can describe any point on L by $\vec{p}_0$ plus a scalar multiple of a direction vector, i.e. $\vec{p}_0 + t\vec{d}$.

Vector equation of a line:

If $\vec{d} \neq \vec{0}$ is a direction vector for a line and $\vec{p}_0$ is a point on the line, then a vector equation for the line is

$$\vec{p} = \vec{p}_0 + t\vec{d}, \quad t \text{ a scalar, or}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}}_{\vec{p}_0} + t \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{d}}$$

Note that choosing a different direction vector and point $\vec{p}_0$ on the line gives a different (but equivalent) description of the line.

Ex: Let L be the line through points $(3,0,-1)$ and $(1,2,1)$.

A direction vector for L is $\vec{d} = \begin{bmatrix} 3-1 \\ 0-2 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$

So an equation for L is

$$\vec{p} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}.$$

We can find points on the line by setting t to different scalars. So e.g. if $t = \frac{-1}{2}$,

$$\vec{p} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3-1 \\ 0+1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Is the point $(9,-6,-7)$ on L?

We need to see if we can find a t so that

$$\begin{bmatrix} 9 \\ -6 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 6 \\ -6 \\ -6 \end{bmatrix} = t \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$, which holds if $t = 3$.

So the point <u>is</u> on L.

Another way to describe L is by breaking it up into coordinates. That is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} x &= 3 + 2t \\ y &= -2t \\ z &= -1 - 2t \end{aligned}\right\} \text{ called parametric equations for } L$$

Ex: Do the lines

$$\vec{p} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \quad \text{and} \quad \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

intersect?

If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ lies on both lines, then

$$\begin{bmatrix} 3+2t \\ -2t \\ -1-2t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2+s \\ 1+3s \end{bmatrix}$$

i.e.

$$\begin{aligned} 3 + 2t &= 1 \\ -2t &= 2+s \\ -1-2t &= 1+3s \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2t &= -2 \\ -2t - s &= 2 \\ -2t - 3s &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & 0 & -2 \\ -2 & -1 & 2 \\ -2 & -3 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ -2 & -1 & 2 \\ -2 & -3 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & -3 & 0 \end{array}\right]$$

$t = -1, \ s = 0.$

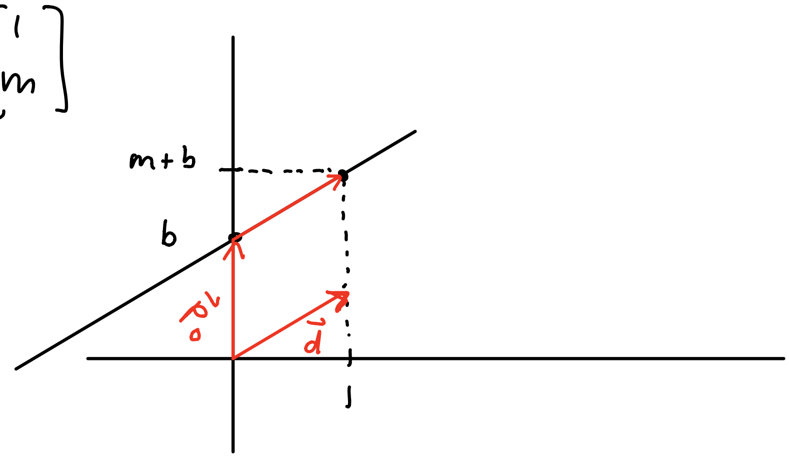
So the lines do intersect, and the point of intersection is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Ex: If $y = mx + b$ is a line in $\mathbb{R}^2$, what is a vector equation for it?

First we find two points on it: $(0, b)$ and $(1, m+b)$.

Set $\vec{d} = \begin{bmatrix} 1-0 \\ (m+b)-b \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}$

and $\vec{p}_0 = \begin{bmatrix} 0 \\ b \end{bmatrix}$.



Then we get

$$\vec{p} = \begin{bmatrix} 0 \\ b \end{bmatrix} + t \begin{bmatrix} 1 \\ m \end{bmatrix}.$$

Parametric equations are

$$x = t$$
$$y = b + mt$$

Ex: The line $y = -2x + 5$ can be written as

$\vec{p} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ or as $\begin{array}{l} x = t \\ y = -2t + 5. \end{array}$

We could also choose a different direction vector, say $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$, and a different $\vec{p}_0$, say $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Then we get a different description of the line:

$$\vec{p} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \end{bmatrix}.$$

Practice problems: 4.1 : 1bf, 3, 4cd, 7bc, 8, 9ae, 10a, 19, 22bdf, 24c